Limits on the detectability of cosmic topology in hyperbolic universes

G.I. Gomero[1]*, M.J. Rebouças[1]†, R. Tavakol[1,2]‡

[1] Centro Brasileiro de Pesquisas Físicas,
Rua Dr. Xavier Sigaud 150
22290-180 Rio de Janeiro – RJ, Brazil

[2] Astronomy Unit, School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road, London E1 4NS, UK

Abstract

We reexamine the possibility of the detection of the cosmic topology in nearly flat hyperbolic Friedmann-Lemaître-Robertson-Walker (FLRW) universes by using patterns repetition. We update and extend our recent results in two important ways: by employing recent observational constraints on the cosmological density parameters as well as the recent mathematical results concerning small hyperbolic 3-manifolds. This produces new bounds with consequences for the detectability of the cosmic topology. In addition to obtaining new bounds, we also give a concrete example of the sensitive dependence of detectability of cosmic topology on the uncertainties in the observational values of the density parameters.

1 Introduction

Within the framework of standard cosmology, the universe seems to be well described by a locally homogeneous and isotropic Robertson-Walker (RW) metric

$$ds^2 = -c^2dt^2 + R^2(t)\,\{\,d\chi^2 + f^2(\chi)\,[\,d\theta^2 + \sin^2\theta\, d\phi^2\,]\,\}\,, \tag{1}$$

where t is a cosmic time, $f(\chi) = \chi\,,\ \ \sin\chi\,,\ \$ or $\sinh\chi\,,\ $ depending on the sign of the constant spatial curvature ($k = 0, \pm 1$), and $R(t)$ is the scale factor. However, a RW metric does not uniquely specify the underlying RW spacetime manifold $\mathcal{M}_4$, which can be decomposed into $\mathcal{M}_4 = \mathcal{R} \times M$. In traditional treatments of cosmology, the 3-space M is usually taken to be one of the following simply-connected spaces: Euclidean E^3, spherical S^3, or hyperbolic space H^3. However, given that the simply-connectedness of our space M has not been established by cosmological observations, our 3-space may equally well be any one of the possible quotient (multiply connected) manifolds $M = \widetilde{M}/\Gamma$, where Γ is a discrete group of isometries of the covering space $\widetilde{M}$ acting freely on $\widetilde{M}$. The action of Γ tessellates the covering space $\widetilde{M}$ into identical cells or domains which are copies of what is known as fundamental polyhedron. An immediate observational consequence of a nontrivial topology (multiple-connectedness) of the 3-space M is that the sky may show patterns repetition, i.e. multiple images of either cosmic objects or spots on the cosmic microwave background radiation, such as circles in the sky.

*gomero@cbpf.br, chapita@gft.ucp.br
†reboucas@cbpf.br
‡r.tavakol@qmwul.ac.uk

Questions of topological nature, such as whether we live in a finite or an infinite universe and what its shape may be are among the fundamental open questions that modern cosmology needs to resolve. These questions go beyond the scope of general relativity (GR), since as a (local) metrical theory GR leaves the global topology of spacetime undetermined.

Given the wealth of increasingly accurate cosmological observations, specially the recent observations of the cosmic microwave background radiation (CMBR [1, 2], these questions have become particularly topical (see, for example, [3] – [5]). It is therefore usually assumed that despite our present-day inability to *predict* the topology of the universe, it will become *detectable* as our observations become more accurate.

An important outcome of the recent observations has been to suggest that the universe is almost flat (see, e.g., [1, 2] and [6] – [8]). This has motivated the recent study of the question of detectability of the cosmic topology in such nearly-flat FLRW universes [9] – [11]. Here we update and extend our works [9, 10] by employing recent observational constraints on the cosmological density parameters as well as the recent mathematical results concerning small hyperbolic 3-manifolds. In addition, we also find a concrete example of sensitive dependence of the detectable set of topologies on the observational bounds on the density parameters.

2 Undetectability Indicators

Regardless of our present-day inability to predict the topology of the universe, its detection and determination is ultimately expected to be an observational problem. Recent studies have however shown that the near-flatness of the universe, deduced from the recent analysis of observations data, may make the task of the detection of a possible nontrivial topology of the universe rather difficult [9] – [11]. More precisely, it has been shown that if one uses patterns repetition, increasing number of nearly flat spherical and hyperbolic possible topologies for the universe become undetectable as $\Omega_0 \to 1$ [9, 10].

The study of the possible non-trivial topology of the spatial sections M requires topological indicators which could be put into correspondence with observations. An intuitive starting point is the comparison between the horizon radius and suitable characteristic sizes of the manifold M. A suitable characteristic size of M, which we shall use in this paper, is the so-called injectivity radius r_{inj} (radius of the smallest sphere 'inscribable' in M), which is defined in terms of the length of the smallest closed geodesics ℓ_M by

$$r_{inj} = \frac{\ell_M}{2} \; . \tag{2}$$

Using r_{inj} we can define the indicator [9]

$$T_{inj} = \frac{r_{inj}}{\chi_{obs}} \; . \tag{3}$$

Now, in any universe for which $T_{inj} > 1$, every source in the survey lies inside a fundamental polyhedron of M, no matter what the location of the observer is within the manifold. As a result there would be no repeated patterns in that survey and every method of the search for cosmic topology based on their existence will fail — the topology of the universe is *undetectable* with this *specific* survey. Now in practice, different surveys may be (and are often) employed. There are three main surveys that can be used in the search for repeated patterns in the universe: namely, clusters of galaxies, containing clusters with redshifts of up to $z_{cluster} \approx 0.3$; active galactic nuclei (mainly QSO's and quasars), with a redshift cut-off of $z_{quasar} \approx 6$; and maps of the CMBR with a redshift of $z_{cmb} \approx 1100$. The crucial point is that the undetectability, based on the employment of the above indicator (3), will be survey dependent. The latter survey, with $z_{cmb} \approx 1100$ corresponding to the redshift of the surface of last scattering, however, has a unique place in practice, as it is in effect a limiting survey with the deepest depth. Thus the quotient (3) computed with z_{cmb} gives the lowest

observational bound in practice for the indicator T_{inj}. At a theoretical level, on the other hand, an absolute lower bound is given by the indicator defined in terms of the horizon radius,

$$T_{inj}^{hor} = \frac{r_{inj}}{\chi_{hor}} \; . \tag{4}$$

The undetectability which arises from the condition $T_{inj}^{hor} \geq 1$ is obviously *survey independent*, and when this inequality holds no multiple images (or patterns repetition) will arise from any survey, including, of course, CMBR. Thus, any method for the search of cosmic topology based on the existence of repeated patterns will fail — the topology of the universe is *definitely undetectable* in such cases.

It is worth emphasizing that the indicator T_{inj} is useful for the identification of cosmological models whose topology is undetectable through methods based on the presumed existence of multiple images, for when $T_{inj} \geq 1$, the whole region covered by a specific survey lies inside a fundamental polyhedron of M. However, without further considerations, nothing can be said about the detectability when $T_{inj} < 1$. In fact, in this case, even if the radius of the depth of a given survey is larger than r_{inj}, it may be that, due to the location of the observer, the whole region covered by the specific survey would still be inside a fundamental polyhedron of M, making the topology undetectable. This is the case when the smallest closed geodesic that passes through the observer is larger than $2\chi_{obs}$.

In section 4 we shall use T_{inj}^{hor} and the indicator T_{inj} to examine the detectability of set of small hyperbolic universes in the light of the most recent observations.

3 Hyperbolic 3-manifolds

In this section we shall briefly recall some relevant facts about hyperbolic 3-manifolds which will be usefull in the following section. We note in passing that in line with the usual mathematical practice in investigations of hyperbolic manifolds, we shall use the curvature radius as the unit of length.

Despite the enormous advances made in the last few decades, there is at present no complete classification of hyperbolic 3-manifolds. However, a number of important results have been obtained. Here we shall briefly recall a number of results concerning closed orientable hyperbolic 3-manifolds which will be useful for our purposes in this work:

1. Mostow's rigidity theorem [12], which ensures a rigid connection between geometrical quantities and topological features in hyperbolic 3-manifolds. Thus once the topology is specified, all metrical quantities, such as the volume and the lengths of their closed geodesics are topological invariants for a given 3-manifold. We note, however, that the volume alone does not uniquely specify the 3-manifold, and consequently there are topologically distinct hyperbolic 3-manifolds with the same volume.

2. Compact orientable hyperbolic 3-manifolds constitute a countable infinity of countably infinite number of sequences, ordered according to their volumes. Moreover, a fixed sequence has an accumulation of compact manifolds near a limiting volume set by a cusped manifold, which has finite volume, is non-compact, and has infinitely long cusped corners [13].

3. According to a result of Thurston [13], there exists a hyperbolic 3-manifold with a minimum volume. This has very recently been shown by Agol [14] to be greater than 0.32095, improving an earlier bound (0.28151) by Przeworski [15];

4. Closed orientable hyperbolic 3-manifolds can be constructed and studied with the publicly available software package SnapPea [16] (see also [17]). The compact manifolds are constructed through a so-called Dehn surgery which is a formal procedure identified by two

coprime integers, i.e. winding numbers (n_1, n_2). SnapPea names manifolds according to the seed cusped manifold and the winding numbers. So, for example, the smallest hyperbolic manifolds is named as $m003(-3, 1)$, where $m003$ corresponds to a seed cusped manifold, and $(-3, 1)$ is a pair of winding numbers.

5. There is a census by Hodgson and Weeks [16, 18] containing 11031 orientable closed hyperbolic 3-manifolds ordered by increasing volumes. Besides the volumes, it also provides other information, such as the solution type, the length of shortest closed geodesic and the first fundamental group. The smallest (volume) manifold in this census (Weeks' manifold) has volume $\mathrm{Vol}(M) = 0.94271$, and is conjectured to be the hyperbolic 3-manifold with minimum volume. But, as was mentioned above, the best current estimate for the volume V of the smallest closed hyperbolic orientable 3-manifold is that in lies in the range $0.32095 < V \leq 0.94271$.

6. Clearly, there is a lower bound on the lengths of geodesics in any finite set of small volume closed orientable hyperbolic 3-manifolds. More importantly, according to a very recent theorem of Hodgson and Kerckhoff [19] the shortest geodesic in closed orientable hyperbolic 3-manifolds with volume less than 1.7011 must have length greater than 0.162, corresponding to a lower bound on r_{inj} of 0.081. We recall that there are 19 manifolds in Hodgson and Weeks census with volume smaller than 1.7011. We also note that the closed census intentionally excludes all manifolds containing geodesics of length less than 0.300, which means that the lower bound of 0.081 may in fact correspond to a larger set of manifolds. It is worth noting that this is an important improvement on the lower bound of 0.09 due to Przeworski [15] which we used in our previous work [10]. That bound was established for a set of manifolds with volumes less than 0.94274, which only contained one known manifold, namely the Weeks' manifold.

4 Detectability and observations

A combination of recent independent astrophysical and cosmological observations seems to indicate that we live in an accelerating FLRW universe with nearly flat spatial sections (with $\Omega_0 \simeq 1$), which contains about $\sim 30\%$ dark matter, close to $\sim 70\%$ dark energy together with a small amount of baryonic matter of the order of few percent (see, for example, references [1, 2] and [6] – [8]).

In the light of these observations, we assume that the universe can be locally described by a FLRW metric (1), and that the matter content of the universe is well approximated by dust of density ρ_m plus a cosmological constant Λ. The Friedmann equation is then given by

$$H^2 = \frac{8\pi G \rho_m}{3} - \frac{kc^2}{R^2} + \frac{\Lambda\, c^2}{3} \,, \tag{5}$$

where $H = \dot{R}/R$ is the Hubble parameter and G is Newton's constant. Introducing $\Omega_m = \frac{8\pi G \rho_m}{3H^2}$ and $\Omega_\Lambda \equiv \frac{8\pi G \rho_\Lambda}{3H^2} = \frac{\Lambda\, c^2}{3H^2}$, and letting $\Omega = \Omega_m + \Omega_\Lambda$, equation (5) gives

$$H^2 R^2 (\Omega - 1) = kc^2 \,. \tag{6}$$

From Eq. (6), for hyperbolic models ($\Omega_0 < 1$), the redshift-distance relation in units of the curvature radius, R_0, reduces to [9]

$$\chi\,(z) = \sqrt{|1 - \Omega_0|} \int_0^z \Big[(1+x)^3 \Omega_{m0} + \Omega_{\Lambda 0} - (1+x)^2 (\Omega_0 - 1) \Big]^{-1/2} dx \,, \tag{7}$$

where the subscript 0 denotes evaluation at present time. The horizon radius χ_{hor} is defined by (7) for $z = \infty$. Written in this form the redshift-distance relation is very convenient for the study of hyperbolic universes, since the curvature radius is used as the unit of length.

To begin with, we recall that the chances of detecting the topology of a nearly flat compact universe from cosmological observations become smaller as $\chi_{hor} \to 0$ ($\chi_{hor} \ll R_0$). Thus as a first step in studying the constraints on detectability we consider the horizon radius function $\chi_{hor}(\Omega_{m0}, \Omega_{\Lambda 0})$ given by (7) with $z = \infty$, for a typical fixed value $\Omega_{m0} = 0.37$, which is the middle value of the bounds on Ω_{m0}, obtained recently [2] by combining measurement of the CMBR anisotropy (BOOMERANG-98, MAXIMA-1 and COBE DMR) together with supernovae Ia (SNIa) and large scale structure (LSS) observations. Figure 1 shows the behaviour of χ_{hor} as a function of $\Omega_{\Lambda 0}$ for this fixed value of Ω_{m0}. Clearly, the limiting case of flat universes ($\Omega_0 = 1$) corresponds to the point at which the curves touch the horizontal axis. This figure clearly demonstrates the rapid way χ_{hor} drops to zero in a narrow neighbourhood of the $\Omega_0 = 1$. From the observational point of view, this shows that the detection of the topology of the nearly flat hyperbolic universes becomes more and more difficult as $\Omega_0 \to 1$, a limiting value favoured by recent observations.

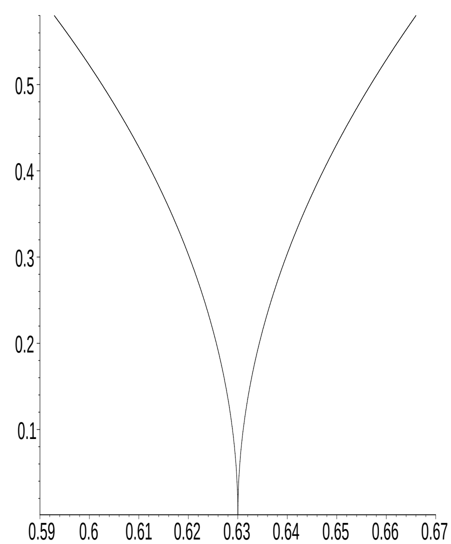

Figure 1: The behaviour of the horizon radius χ_{hor} in units of curvature radius, for FLRW models with ρ_m and Λ as a function of the density parameters Ω_Λ for $\Omega_m = 0.37$, which is the middle value for Ω_m. This figure shows clearly the rapid way χ_{hor} falls off to zero for nearly flat hyperbolic universes, as $\Omega_0 = \Omega_{m0} + \Omega_{\Lambda 0} \to 1$. The vertical represents χ_{hor}, while the horizontal axis gives Ω_Λ.

To obtain more quantitative information, we employ the indicator T_{inj} to examine the detectability of cosmic topology of hyperbolic universes with nontrivial topologies. Given the existence of more recent estimates of the cosmological density parameters, we shall update and extend our previous results by considering, in addition to the hyperbolic sub-interval given by Bond *et al.* [20]:

$$\Omega_0 \in [0.99, 1) \qquad \text{and} \qquad \Omega_{\Lambda 0} \in [0.63, 0.73] \tag{8}$$

the hyperbolic sub-interval consistent with a more recently bound on the density parameters given by Jaffe *et al.* [2]

$$\Omega_0 \in [0.98, 1) \qquad \text{and} \qquad \Omega_{\Lambda 0} \in [0.62, 0.79]\ . \tag{9}$$

To make a comparative study, we consider each set of these bounds in turn. Using the hyperbolic sub-interval (8), one can calculate from (7) the largest values of χ_{obs} in this interval. For $z_{max} = 6$ one finds $\chi_{obs}^{max} = 0.20125$, while for $z_{max} = 1100$ (CMBR) one finds $\chi_{obs}^{max} = 0.33745$. Thus, using quasars up to $z_{max} = 6$, FLRW hyperbolic universes with the density parameters in (8) have undetectable topologies if their corresponding injectivity radii are such that $r_{inj} \geq 0.20125$. Similarly, for the same hyperbolic sub-interval, using CMBR, the topology of hyperbolic universes with $r_{inj} \geq 0.33745$ is undetectable. Further, for $z_{max} = \infty$, the largest value of χ_{obs} in the

sub-interval (8) is $\chi_{hor} = 0.349247$, so the topology of hyperbolic universes with $r_{inj} \geq 0.34924$ is definitely undetectable regardless of depth of the survey. In Table 1 we have summarized the restrictions on detectability imposed by the hyperbolic sub-interval (8) on the first seven manifolds of Hodgson-Weeks census, where U denotes that the topology is undetectable by any survey of depth up to the redshifts $z_{max} = 6$ (quasars) or $z_{max} = 1100$ (CMBR) respectively. Thus using quasars, the topology of the five known smallest hyperbolic manifolds, as well as m009(4,1), are undetectable within the hyperbolic region of the parameter space given by (8), while only topologies m007(3,1) and m009(4,1) remain undetectable even if CMBR observations are used. This shows clearly how detectability depends concretely on the survey used.

M	r_{inj}	QUASARS	CMBR
m003(-3,1)	0.292	U	—
m003(-2,3)	0.289	U	—
m007(3,1)	0.416	U	U
m003(-4,3)	0.287	U	—
m004(6,1)	0.240	U	—
m004(1,2)	0.183	—	—
m009(4,1)	0.397	U	U

Table 1: Restrictions on detectability of cosmic topology originated from the sub-interval (8) for the first seven manifolds of Hodgson-Weeks census. Here U stands for undetectable using catalogues of quasars (up to $z_{max} = 6$) or CMBR ($z_{max} = 1100$).

Considering now the second hyperbolic sub-interval (9), one can again calculate from (7) the largest values of χ_{obs} in this interval: for $z_{max} = 6$ one has $\chi_{obs}^{max} = 0.313394$, while for $z_{max} = 1100$ (CMBR) one finds $\chi_{obs}^{max} = 0.538276$. Thus, using quasars up to $z_{max} = 6$, FLRW hyperbolic universes with the density parameters in (9) have undetectable topologies if $r_{inj} \geq 0.313394$. Similarly, for the same hyperbolic sub-interval, using CMBR, the topology of hyperbolic universes with $r_{inj} \geq 0.538276$ is undetectable. Further, for $z_{max} = \infty$, the largest value of χ_{obs} in the sub-interval (8) is $\chi_{hor} = 0.557832$, so the topology of hyperbolic universes with r_{inj} greater than this value is definitely undetectable regardless of the depth of the survey. The restrictions on detectability imposed by the hyperbolic sub-interval (9) on the first seven manifolds of Hodgson-Weeks census can again be reexamined. Using this sub-interval we find a very different picture from that summarized in Table 1, namely that in this case, using quasars, only two topologies [$m007(3,1)$ and $m009(4,1)$] would be undetectable whereas using CMBR none of topologies of these seven manifolds (universes) would be undetectable. Table 2 summarizes the restrictions on detectability imposed by the hyperbolic sub-interval (9) on the first seven manifolds of Hodgson-Weeks census.

The results in Table 2 together with those in Table 1 make transparent that a variation of 1% in the total density parameters Ω_0 ($0.99 \to 0.98$), for $\Omega_{\Lambda 0} \in [0.62, 0.79]$, which would have no significant consequences in the geometrical (dynamical) features of the universes, would crucially change the detectability of cosmic topology.

One can also reexamine, in the light of the new bounds (9), what is the region of the parameter space for which a given set of topologies are undetectable. To this end we note that for a given topology (fixed r_{inj}) and for a given survey up to z_{max}, one can solve the equation

$$\chi_{obs}(\Omega, \Omega_\Lambda) \,=\, r_{inj} \;, \tag{10}$$

which amounts to finding pairs (Ω, Ω_Λ) in the density parameter plane for which Eq. (10) holds.

Consider now the set of the 19 smallest manifolds of the Hodgson-Weeks census in conjuction with the hyperbolic region (9) and the eqs. (7) and (10). The manifold in this set with the lowest $r_{inj}(= 0.152)$ is $m003(-5,4)$. Figure 2 gives the solution curve of equation (10) in the $\Omega_0 - \Omega_{\Lambda 0}$

M	r_{inj}	QUASARS	CMBR
m003(-3,1)	0.292	—	—
m003(-2,3)	0.289	—	—
m007(3,1)	0.416	U	—
m003(-4,3)	0.287	—	—
m004(6,1)	0.240	—	—
m004(1,2)	0.183	—	—
m009(4,1)	0.397	U	—

Table 2: Restrictions on detectability of cosmic topology originated from the sub-interval (9) for the first seven manifolds of Hodgson-Weeks census. Here U stands for undetectable using catalogues of quasars (up to $z_{max} = 6$) or CMBR ($z_{max} = 1100$).

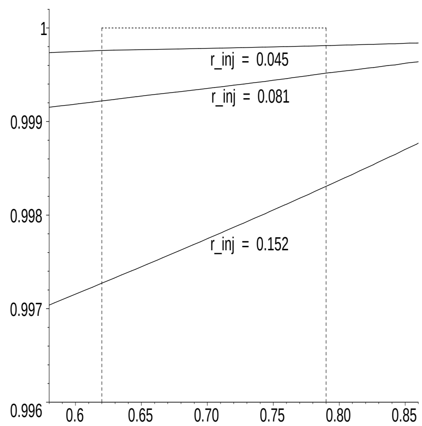

Figure 2: The solution curves of $\chi_{obs} = r_{inj}$, as plots of Ω_0 (vertical axis) versus $\Omega_{\Lambda 0}$ (horizontal axis), for $r_{inj} = 0.045$ (upper curve),$r_{inj} = 0.081$ and $r_{inj} = 0.152$ (lower curve). A survey with depth $z_{max} = 1100$ (CMBR)was used in both cases. The dashed rectangular box represents the relevant part, for our purposes, of the hyperbolic region (9) of the parameter space given by recent observations. The undetectable regions of the parameter space $(\Omega_0, \Omega_{\Lambda 0})$, corresponding to each value of r_{inj}, lie above the related curve.

plane for $r_{inj} = 0.152$ and $r_{inj} = 0.081$, where a survey of depth $z_{max} = 1\,100$ (CMBR) was used.[1] This figure also contains a dashed rectangular box, representing the relevant part (for our purposes here) of the recent hyperbolic region (9). For each value of r_{inj} undetectability is ensured for the values of cosmological parameters (region in the $\Omega_0 - \Omega_{\Lambda 0}$ plane) which lie above the corresponding solution curve of (10). Thus considering the solution curve of (10) for $r_{inj} = 0.081$, one finds that all closed orientable hyperbolic manifolds (universes) with volumes less than 1.0711, for example, would have undetectable topology, if the total density Ω_0 turned out to be higher than ~ 0.9994. Similarly, considering the solution curve of (10) for $r_{inj} = 0.152$, for example, one finds that the topology of none of the 19 smallest manifolds of the census would be detectable, if Ω_0 turned out to be higher than ~ 0.9974.

[1] This figure updates Figure 1 of in [10] two regards: first it employs a more recent hyperbolic sub-interval of the cosmological density parameters [2]; second it uses the most recent lower bound on the length of shortest closed geodesic in closed orientable hyperbolic 3-manifolds [19]. Note that as opposed to the manifolds considered in [10] which could contain only Weeks' manifold, now there are at least 19 manifolds.

5 Conclusions

Motivated by the recent observational results indicating that the universe is nearly flat, we have employed the recent analyses of the observational constraints on the cosmological density parameters, together with recent mathematical results concerning small hyperbolic 3-manifolds, to examine the possibility of detecting the topology of nearly flat hyperbolic universes by using patterns repetition. In this way we have updated and extended our recent results, which has resulted in new bounds on detectable topologies.

In addition we have also found that small changes in the cosmological density parameters of the order a few percent are sufficient to radically effect the detectability of the topology of small hyperbolic universes. This result, which is essentially the consequence of the rapid way the horizon radius χ_{hor} falls off to zero for nearly flat hyperbolic universes, is of great potential importance, as it demonstrates concretely how small changes in the observational bounds on the cosmological density parameters could have important consequences for the question of detectability of the cosmic topology.

Acknowledgments

We are grateful to Ian Agol and Andrew Przeworski for very helpful correspondence concerning their work, and for drawing our attention to the [19]. We also thank CNPq, MCT/CBPF and CLAF for the grants under which this work was carried out.

References

[1] P. de Bernadis *et al.*, *Nature* **404**, 955 (2000);
S. Hanany *et al.*, *Astrophys. J. Lett.* **545**, 5 (2000);
A.E. Lange *et al.*, *Phys. Rev. D* **63**, 042001 (2001);
P. de Bernardis *et al.*, *First Results from BOOMERANG Experiment*, astro-ph/0011469 (2000). In Proc. of the CAPP2000 conference, Verbier, 17-28 July 2000;
J.R. Bond *et al.*, *The Cosmic Background Radiation circa ν2K*, astro-ph/0011381 (2000). In Proc. of Neutrino 2000 (Elsevier), CITA-2000-63, Eds. J. Law & J. Simpson;
J.R. Bond *et al.*, *The Quintessential CMB, Past & Future*. In Proc. of CAPP-2000 (AIP), CITA-2000-64;
J.R. Bond *et al.*, *CMB Analysis of Boomerang & Maxima & the Cosmic Parameters*, astro-ph/0011378 (2000). In Proc. IAAU Symposium 201 (PASP), CITA-2000-65;
A. Balbi *et al.*, *Astrophys. J.* **545**, L1–L4 (2000).

[2] A.H. Jaffe *et al.*, *Phys. Rev. Lett.* **86**, 3475 (2001).

[3] G.F.R. Ellis, *Gen. Rel. Grav.* **2**, 7 (1971);
D.D. Sokolov & V.F. Shvartsman, *Sov. Phys. JETP* **39**, 196 (1974);
G.F.R. Ellis & G. Schreiber, *Phys. Lett. A* **115**, 97 (1986);
R. Lehoucq, M. Lachièze-Rey & J.-P. Luminet, *Astron. Astrophys.* **313**, 339 (1996);
B.F. Roukema, *Mon. Not. R. Astron. Soc.* **283**, 1147 (1996);
G.F.R. Ellis & R. Tavakol, *Class. Quantum Grav.* **11**, 675 (1994);
J.J. Levin, J.D. Barrow & J. Silk, *Phys. Rev. Lett.* **79**, 974 (1997);
N.J. Cornish, D.N. Spergel & G.D. Starkman, *Class. Quantum Grav.* **15**, 2657 (1998);
N.J. Cornish, D.N. Spergel & G.D. Starkman, *Proc. Nat. Acad. Sci.* **95**, 82 (1998);
N.J. Cornish, D. Spergel & G. Starkman, *Phys. Rev. D* **57**, 5982 (1998);
J.J. Levin, E. Scannapieco & J. Silk, *Phys. Rev. D* **58**, 103516 (1998);
J.J. Levin, E. Scannapieco & J. Silk, *Class. Quantum Grav.* **15**, 2689 (1998);

J.J. Levin, E Scannapieco, E. Gasperis, J. Silk, & J.D. Barrow, *Phys. Rev. D* **58**, 123006 (1999);
R. Lehoucq, J.-P. Luminet & J.-P. Uzan, *Astron. Astrophys.* **344**, 735 (1999);
J.-P. Uzan, R. Lehoucq & J.-P. Luminet, *Astron. Astrophys.* **351**, 776 (1999);
H.V. Fagundes & E. Gausmann, *Phys. Lett. A* **261**, 235 (1999);
R. Aurich, *Astrophys. J.* **524**, 497 (1999);
J.R. Bond, D. Pogosyan & T. Souradeep, *Phys. Rev. D* **62**, 043005 (2000);
J.R. Bond, D. Pogosyan & T. Souradeep, *Phys. Rev. D* **62**, 043006 (2000);
G.I. Gomero, M.J. Rebouças & A.F.F. Teixeira, *Phys. Lett. A* **275**, 355 (2000);
G.I. Gomero, M.J. Rebouças & A.F.F. Teixeira, *Int. J. Mod. Phys. D* **9**, 687 (2000);
G.I. Gomero, M.J. Rebouças & A.F.F. Teixeira, *Class. Quantum Grav.* **18**, 1885 (2001);
M.J. Rebouças, *Int. J. Mod. Phys. D* **9**, 561 (2000);
G.I. Gomero, A.F.F. Teixeira, M.J. Rebouças & A. Bernui, *Int. J. Mod. Phys. D* **11**, 869 (2002).
G.I. Gomero & M.J. Rebouças, *Detectability of Cosmic Topology in Flat Universes*, gr-qc/0203094 (2002).
R. Aurich & F. Steiner, *Mont. Not. Roy. Astron. Soc.* **323**, 1016 (2001).

[4] J.J. Levin, *Phys. Rep.* **365**, 251 (2002).
R. Lehoucq, J.-P. Uzan & J.-P. Luminet, astro-ph/0005515 (2000);
V. Blanlœil & B.F. Roukema, Eds., astro-ph/0010170 (2000);
G.D. Starkman, *Class. Quantum Grav.* **15**, 2529 (1998);
M. Lachièze-Rey & J.-P. Luminet, *Phys. Rep.* **254**, 135 (1995).

[5] Ya. B. Zeldovich & I.D. Novikov, *The Structure and Evolution of the Universe*, p. 633-640, The University of Chicago Press (1983). See on p. 637 refs. of the earlier works by Süveges (1966), Sokolov (1970), Paál (1971), Sokolov and Shvartsman (1974), and Starobinsky (1975).

[6] B.P. Schmidt *et al.*, *Astrophys. J.* **507** 46 (1998); A.G. Riess *et al.*, *Astron. J.* **116**, 1009 (1998).

[7] B.F. Roukema, G.A. Mamon & S. Bajtlik, *Astron. Astrophys.* **382**, 397 (2002).

[8] S. Perlmutter *et al.*, *Astrophys. J.* **517**, 565 (1999); S. Perlmutter, M.S. Turner & M. Write, *Phys. Rev. Lett.* **83**, 670 (1999).

[9] G.I. Gomero, M.J. Rebouças & R. Tavakol, *Class. Quantum Grav.* **18**, 4461 (2001).

[10] G.I. Gomero, M.J. Rebouças & R. Tavakol, *Class. Quantum Grav.* **18**, L145 (2001).

[11] E. Gausmann, R. Lehoucq, J-P Luminet, J-P Uzan, J. Weeks, *Class. Quantum Grav.* **18**, 5155 (2001).

[12] G.D. Mostow, *Ann. Math. Studies* **78** (1973), Princeton University Press, Princeton, New Jersey.

[13] W.P. Thurston, *Bull. Am. Math. Soc.* **6**, 357 (1982).

[14] I. Agol *Volume Change under Drilling*, preprint available at `http://xxx.lanl.gov/abs/math.GT/0101138` (2001).

[15] A. Przeworski, *J. Differential Geom.* **58**, 2 (2001). Also available at `http://www.ma.utexas.edu/users/prez/`.

[16] J.R. Weeks, SnapPea: A computer program for creating and studying hyperbolic 3-manifolds, available at `http://thames.northnet.org/weeks/`

[17] C. Adams, *Not. Am. Math. Soc.* **37**, 273 (1990).

[18] C.D. Hodgson & J.R. Weeks, *Experimental Mathematics* **3**, 261 (1994).

[19] C.D. Hodgson & Steven P. Kerckhoff, *Universal Bounds for Hyperbolic Dehn Surgery*, preprint available at http://xxx.lanl.gov/abs/math.GT/0204345.(2002).

[20] J.R. Bond *et al.*, *The Quintessential CMB, Past & Future*. In Proc. of CAPP-2000 (AIP), CITA-2000-64.